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A class of second-order differential equations and related first-order systems

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Abstract. A class of second-order non-linear differential equations which arises in several branches of mathematical physics is considered. It is shown that equations of this class may be factorised into first-order equations of 'Riccati type'. Conditions are obtained, on the coefficient functions of the second-order equations, for the first-order equations to be of matrix Riccati form, whose solutions have a finite superposition property. The factorisation into first-order equations is then not unique, and there is an alternative first-order set of equations whose solutions do not have this superposition property.

A second-order equation arising in the theory of pellet fusion processes is investigated in detail. Solutions are obtained when the corresponding first-order equations are of matrix Riccati from and shown to be equivalent to solutions derived by alternative methods. Lagrangian systems giving rise to equations of the class are also considered.

1. Introduction

A class of non-linear ordinary differential equations of particular interest in mathematical physics is defined by the equation

$$y''(x)+(E_0(x)+E_1(x)y(x))y'(x)+F_0(x)+F_1(x)y(x)+F_2(x)(y(x))^2 + F_3(x)(y(x))^3=0 \tag{1.1}$$

where E_0, E_1, F_0, F_1, F_2, F_3 are given functions of the independent variable x. Four particular examples are as follows.

(i) The scalar field equation in one-dimensional φ^4 field theory

$$y''-y-ky^3=0 \tag{1.2}$$

where k is a constant.

(ii) The equation

$$y''+(3\gamma+1)yy'+(3\gamma-1)y^3+c=0 \tag{1.3}$$

where γ, c are constant, which arose in the study of the pellet fusion process by Ervin *et al* (1984).

(iii) An equation governing spherically symmetric expansion or collapse of a relativistically gravitating mass derived by McVittie (1933, 1967, 1984) which is of the form

$$y''+(c+y)y'-(d+cy+y^2)y=0 \tag{1.4}$$

where c, d are constants.

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(iv) An equation which can be thought of as a one-dimensional analogue of the boson 'gauge theory' equations introduced by Yang and Mills (1954). The equation is of the form

$$[\mathrm{d}/\mathrm{d}x+f(x)+g(x)y][\mathrm{d}/\mathrm{d}x+f(x)+g(x)y]y+k(x)+h(x)y=0 \tag{1.5}$$

and is equivalent to (1.1) with

$$E_0=2f \qquad E_1=3g \qquad F_0=k \qquad F_1=f'+f^2+h$$
$$F_2=g'+2fg \qquad F_3=g^2. \tag{1.6}$$

Our investigation was initially stimulated by the work of McVittie (1933, 1984), who obtained solutions of (1.4) which were also solutions of the Riccati equations. The generalised derivative

$$D_x \equiv \mathrm{d}/\mathrm{d}x+f(x)+g(x)y \tag{1.7}$$

occurring in (1.5) is termed the 'Riccati operator', since the Riccati equation can be written

$$D_x y+h(x)=y'+h+fy+gy^2=0. \tag{1.8}$$

The Riccati operator is linear in $\mathrm{d}/\mathrm{d}x$ and y and we say that (1.8) is of 'index 1'. Likewise, since (1.1) is formed by the quadratic action of $\mathrm{d}/\mathrm{d}x$ and y on y itself, we say that this equation has 'index 2'.

In this paper, we investigate the equivalence of equations of the general form (1.1) to a pair of coupled equations of 'Riccati type'

$$y'=A_1+B_1y+B_2z+D_1y^2+D_2yz+D_5z^2 \tag{1.9a}$$

$$z'=A_2+B_3y+B_4z+D_3y^2+D_4yz+D_6z^2 \tag{1.9b}$$

where the coefficients A_i, B_i, D_i are, in general, functions of x.

In § 2 we show that (1.1) can always be written in the form (1.9) and that this form is *not* unique. This is to be expected since there are twelve independent coefficients $\{A_i, B_i, D_i\}$ in (1.5) while there are only six in (1.1).

Lie studied systems of n first-order non-linear equations (Lie and Engel 1893, Lie and Scheffers 1893, see also Hermann and Ackermann 1973), of which (1.9) is a particular example. He investigated the conditions for the general solution of a system to be expressible in terms of a finite number of particular solutions. He showed that the equations have this 'superposition' property if they are associated with a finite-dimensional Lie algebra. In recent years, numerous authors have studied the classification of those non-linear differential equations which are associated with the classical Lie algebras. (A selection of references is given by Winternitz (1983).) In § 3 we investigate when (1.9) belongs to one of these classes. We shall show that if the coefficients in the original index 2 equation (1.1) satisfy the conditions

$$9F_3=E_1^2 \tag{1.10a}$$

$$3F_2=E_1'+E_0E_1 \tag{1.10b}$$

then an equivalent set of index 1 equations can be related to the Lie algebra of the projective group $\mathrm{PL}(n, \boldsymbol{R})$ with $n=2$. The equation may then be written in the matrix Riccati form

$$W'=A+BW+WC+WDW \tag{1.11}$$

where

$$W = \begin{pmatrix} y(x) \\ z(x) \end{pmatrix} \tag{1.12}$$

contains the unknown functions and

$$A = \begin{pmatrix} A_1(x) \\ A_2(x) \end{pmatrix} \qquad B = \begin{pmatrix} B_1(x) & B_2(x) \\ B_3(x) & B_4(x) \end{pmatrix} \tag{1.13}$$

$$C = C(x) \qquad D = (D_1(x) D_2(x))$$

consists of given functions. The function C can be absorbed into B_1 and B_4, and for (1.9) to be a 'Lie system' with an associated finite-dimensional group, the D_i in (1.9) are not independent. The form (1.11) is particularly useful since it may be integrated at least formally, as we show in § 3.

In their study of (1.3), Ervin *et al* (1984) demonstrated that solutions in terms of trigonometrical functions exist for $\gamma = \frac{5}{3}, \frac{2}{3}$ and $\frac{1}{3}$. We will show in § 4 that the values $\gamma = \frac{5}{3}, \frac{2}{3}$ are precisely the values when (1.9) become a Lie system; the equations can then be solved exactly. The third root $\gamma = \frac{1}{3}$ corresponds to a soluble degenerate set (1.9).

In the final section, we present our conclusions on this work and suggest further areas of study.

2. Factorisation of the index 2 equation

If y, z satisfy (1.9) and we define

$$w(x) = D_6 y(x) - D_5 z(x)$$

then it is easy to show that $w(x)$ satisfies an equation of the form (1.9a) with D_5 identically zero. Therefore, we can take our index 1 equations to be

$$y' = A_1 + B_1 y + B_2 z + D_1 y^2 + D_2 yz \tag{2.1a}$$

$$z' = A_2 + B_3 y + B_4 z + D_3 y^2 + D_4 yz + D_6 z^2. \tag{2.1b}$$

Then from (2.1a)

$$z = (y' - A_1 - B_1 y - D_1 y^2)/(B_2 + D_2 y). \tag{2.2}$$

(Note that if we had used (1.9a) with $D_5 \neq 0$ we would have had to solve a quadratic equation for z in terms of y and y'.) Substituting for z in (2.1b) we obtain the second-order differential equation for y:

$$\begin{aligned}(y'' - A_1' - B_1' y - B_1 y' - D_1' y^2 - 2D_1 yy')(B_2 + D_2 y)& \\ -(y' - A_1 - B_1 y - D_1 y^2)(B_2' + D_2' y + D_2 y')& \\ = (A_2 + B_3 y + D_3 y^2)(B_2 + D_2 y)^2& \\ + (B_4 + D_4 y)(B_2 + D_2 y)(y' - A_1 - B_1 y - D_1 y^2)& \\ + D_6 (y' - A_1 - B_1 y - D_1^2 y^2)^2.&\end{aligned} \tag{2.3}$$

In general, if this equation is to be of the form (1.1) we must choose either

$$D_1 \equiv D_2 \equiv 0 \tag{2.4a}$$

or

$$D_2 \equiv D_6 \equiv 0. \tag{2.4b}$$

The choice (2.4a) reduces (2.1a) to a linear equation.

We will confine our study to the choice (2.4b), so that both of equations (2.1) are of true 'Riccati form'. In this case, comparing (2.3) with (1.1) gives

$$E_0 = -B_1 - B_4 - B_2'/B_2 \tag{2.5a}$$

$$E_1 = -2D_1 - D_4 \tag{2.5b}$$

$$F_0 = A_1 B_4 - A_1' + A_1 B_2'/B_2 - A_2 B_2 \tag{2.5c}$$

$$F_1 = A_1 D_4 + B_1 B_4 - B_1' + B_1 B_2'/B_2 - B_2 B_3 \tag{2.5d}$$

$$F_2 = B_1 D_4 - B_2 D_3 + B_4 D_1 - D_1' + D_1 B_2'/B_2 \tag{2.5e}$$

$$F_3 = D_1 D_4 \tag{2.5f}$$

provided that $B_2(x) \neq 0$ in the range considered.

Given then the six coefficients in (1.1), we can fit them by choosing the nine functions $\{A_1, A_2, B_1, B_2, B_3, B_4, D_1, D_3, D_4\}$ in (2.5). Three of these can be chosen arbitrarily, and we take them to be $A_1(x)$, $B_1(x)$ and $B_2(x)$, with $B_2 \neq 0$. Then (2.5b) and (2.5f) can be solved giving two possible choices of the coefficients D_4, D_1 of quadratic terms:

$$D_4 = -\tfrac{1}{2}[E_1 \pm (E_1^2 - 8F_3)^{1/2}] \tag{2.6a}$$

$$D_1 = -\tfrac{1}{4}[E_1 \mp (E_1^2 - 8F_3)^{1/2}]. \tag{2.6b}$$

Then B_4, A_2, B_3 and D_3 are in turn determined by (2.5a), (2.5c), (2.5d) and (2.5e). Note that if we had chosen a set of arbitrary coefficients different from $\{A_1, B_1, B_2\}$, we would have had to solve a differential equation to obtain some of the remaining coefficients. If we make the particular choice $B_1(x) \equiv b$, $B_2(x) \equiv 1$, $A_1(x) \equiv 0$, where b is independent of x, we obtain the following simple expressions for the remaining coefficients:

$$\begin{gathered} B_4 = -E_0 - b \qquad A_2 = -F_0 \qquad B_3 = -F_1 - b(E_0 + b) \\ D_3 = -F_2 - D_1' + bD_4 - (b + E_0)D_1. \end{gathered} \tag{2.7}$$

These relations are simplified when $b = 0$, but we will see in § 4 that it is advantageous in some situations to choose a non-zero value for b. It is important to note that even when $\{A_1, B_1, B_2\}$ have been chosen, there is still a *twofold* ambiguity in (1.9) corresponding to the choice of sign in (2.6).

3. Equations of Lie type

With the particular choice of coefficients in (1.9) made in the previous section, these equations have the form

$$y' = by + z + D_1 y^2 \tag{3.1a}$$

$$z' = A_2 + B_3 y + B_4 z + D_3 y^2 + D_4 yz. \tag{3.1b}$$

The condition for the solutions of these equations to possess a superposition property is governed by Lie's fundamental result (Lie and Engel 1893, Lie and Scheffers 1893, see also Herrmann and Ackermann 1973) as follows.

Theorem. The general solution of the system of non-linear first-order differential equations

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}x} = \eta^\mu(\boldsymbol{u}, x) \qquad \mu = 1, \ldots, N \tag{3.2}$$

can be expressed as a superposition of a finite number m of particular solutions $\boldsymbol{u}_{(1)}(x), \ldots, \boldsymbol{u}_{(m)}(x)$ if and only if (3.2) can be written in the form

$$\frac{\mathrm{d}u^\mu}{\mathrm{d}x} = \sum_{k=1}^{r} Z_k(x)\xi_k^\mu(\boldsymbol{u}) \tag{3.3}$$

where the operators

$$X_k = \sum_{\mu=1}^{N} \xi_k^\mu(\boldsymbol{u}) \frac{\partial}{\partial u^\mu} \qquad k = 1, \ldots, r \tag{3.4}$$

generate a finite-dimensional Lie algebra. Then

$$[X_a, X_b] = \sum_{c=1}^{r} C_{ab}^c X_c \tag{3.5}$$

where the C_{ab}^c are the structure constants of the algebra.

If our equations (3.1) are to be of the form (3.3), the corresponding elements X_k of the Lie algebra will contain the operators ∂_z, $y\partial_y$, $z\partial_y$, $y\partial_z$, $z\partial_z$, $y^2\partial_y$, $y^2\partial_z$ and $yz\partial_z$ where $\partial_y = \partial/\partial y$ and $\partial_z = \partial/\partial z$. The Lie algebras corresponding to the general Riccati equation (1.9) with two dependent variables were originally obtained by Lie and Engel (1893) and have been studied more recently by Hlavaty *et al* (1984). They found that the algebras associated with (1.9) contain at most *two* independent operators $P(y, z)\partial y$, $Q(y, z)\partial_z$ with $P(y, z)$, $Q(y, z)$ quadratic functions of y, z. The algebras belong to two classes.

(a) The quadratic operators are

$$\begin{aligned} X_1 &= y^2\partial_y + yz\partial_z \\ X_2 &= yz\partial_z + z^2\partial_z. \end{aligned} \tag{3.6}$$

(b) The quadratic operators are

$$\begin{aligned} X_1 &= \alpha y^2\partial_y + 2\beta yz\partial_y + \gamma y^2\partial_z + \beta z^2\partial_z \\ X_2 &= \gamma y^2\partial_y + \beta z^2\partial_y + 2\gamma yz\partial_z - \alpha z^2\partial_z \end{aligned} \tag{3.7}$$

where α, β and γ are arbitrary constants satisfying $|\alpha|+|\beta|+|\gamma|>0$.

Let us first consider when an algebra of class b gives the quadratic terms in (3.1). Setting $u_1 = y$, $u_2 = z$ in (3.3) and (3.4), equations (3.3) become

$$\begin{aligned} y' &= Z_1(x)(\alpha y^2 + 2\beta yz) + Z_2(x)(\gamma y^2 + \beta z^2) + \mathrm{O}(y, z) \\ z' &= Z_1(x)(\gamma y^2 + \beta z^2) + Z_2(x)(2\gamma yz - \alpha z^2) + \mathrm{O}(y, z). \end{aligned} \tag{3.8}$$

These are equivalent to (3.1) only if

$$2\beta Z_1(x) = 0 \qquad \beta Z_2(x) = 0 \qquad \beta Z_1(x) = Z_2(x)\alpha. \tag{3.9}$$

If both Z_1 and Z_2 are identically zero, there are no quadratic terms in (3.1a, b), which then reduce to a pair of coupled linear differential equations. We are interested in cases when (3.1) contains quadratic terms. There are two corresponding solutions of (3.9):

$$\alpha=\beta=0 \qquad Z_i \not\equiv 0 \tag{3.10}$$

and

$$\beta=0 \qquad Z_2(x)\equiv 0 \qquad Z_1\not\equiv 0. \tag{3.11}$$

The solution (3.10) gives the complete algebra

$$\mathscr{L}_2=\{y^2\partial_z,\ y^2\partial_y+2yz\partial_z,\ y\partial_y,\ y\partial_z,\ z\partial_z,\ z\partial_z,\ \partial_y,\ \partial_x\}. \tag{3.12}$$

The corresponding equations (3.8) are then

$$y'=Z_6(x)+Z_3(x)y+0+Z_2(x)y^2 \tag{3.13a}$$

$$z'=Z_7(x)+Z_4(x)y+Z_5(x)z+Z_1(x)y^2+2Z_2(x)yz \tag{3.13b}$$

where the $Z_i(x)$ are the multipliers of the elements of $\mathscr{L}_2$ given in (3.12), numbering from left to right. Note that because of the absence of a $z\partial_y$ operator in $\mathscr{L}_2$ there is no term linear in z on the right of (3.13a). Therefore $\mathscr{L}_2$ defined by (3.12) cannot be associated with the equations (3.1).

For solution (3.11) and with $\alpha^2+4\beta\gamma\neq 0$, the complete algebra is

$$\mathscr{L}_2=\{\alpha y^2\partial_y+\gamma y^2\partial_z,\ \gamma y^2\partial_y+2\gamma yz\partial_z-\alpha z^2\partial_z,\ y\partial_y+z\partial_z,\ \alpha(y\partial_y-z\partial_z)+2\gamma y\partial_z,\ \partial_y,\ \partial_z\}. \tag{3.14}$$

This six-dimensional algebra is isomorphic to O(2, 2) or O(3, 1), depending on the sign of $\alpha^2+4\beta\gamma$ (Havlicek and Lassner 1975). Remembering that in this case the multiplier $Z_2=0$, the corresponding differential equations are

$$y'=Z_5(x)+(Z_3(x)+\alpha Z_4(x))y+0+\alpha Z_1(x)y^2 \tag{3.15a}$$

$$z'=Z_6(x)+2\gamma Z_4(x)y+(Z_3(x)-\alpha Z_4(x))z+\gamma Z_1(x)y^2. \tag{3.15b}$$

Once again there is no term linear in z on the RHS of (3.15a) and so $\mathscr{L}_2$ defined by (3.14) cannot be associated with (3.1). Equations (3.13) and (3.15) are said to be 'decomposable', while we concentrate here on 'indecomposable' Lie systems of equations. These concepts have been introduced recently by Shnider and Winternitz (1984).

We must therefore examine class a, with quadratic operators given in (3.6); these are contained in the eight-dimensional Lie algebra

$$\mathscr{L}_1=\{y^2\partial_y+yz\partial_z,\ yz\partial_z+z^2\partial_z,\ y\partial_y,\ y\partial_z,\ z\partial_y,\ z\partial_z,\ \partial_y,\ \partial_z\}$$

which is isomorphic to sl(3, R). In the usual notation the corresponding differential equations are

$$y'=Z_1(x)y^2+Z_2(x)yz+\mathrm{O}(y,z) \tag{3.16a}$$

$$z'=Z_1(x)yz+Z_2(x)z^2+\mathrm{O}(y,z). \tag{3.16b}$$

These are equivalent to (3.1) if

$$Z_2(x)\equiv 0 \tag{3.17a}$$

and

$$D_1(x)\equiv D_4(x) \qquad D_3(x)\equiv 0. \tag{3.17b}$$

Equations (3.6) and (3.7) define the only classes of $N=2$ Lie algebras with *two* independent quadratic operators. These are also algebras with just one quadratic operator. Hlavaty *et al* (1984) were not able to classify them; we have investigated them and have found that they do not lead to non-trivial examples of (3.1). So the only Lie algebra corresponding to non-trivial (3.1) is $\mathscr{L}_1$, imposing the conditions (3.17*b*) on the coefficients and the condition $Z_2 \equiv 0$ upon (3.16). There is therefore no exact correspondence between the Lie algebra and equations (3.1).

The conditions (3.17*b*) imply conditions on the coefficients of our original non-linear differential equation (1.1) for the existence of a corresponding system (3.1) with a superposition property. From (2.6), these conditions are

$$2[E_1 \pm (E_1^2 - 8F_3)^{1/2}] = E_1 \mp (E_1^2 - 8F_3)^{1/2}$$

or, on squaring,

$$E_1^2 = 9F_3 \tag{1.10a}$$

and

$$3F_2 = E_1' + E_0 E_1 \tag{1.10b}$$

as stated in § 1. We note that if (1.10*a*) is satisfied, the two solutions (2.6) become

$$D_1 = D_4 = -\tfrac{1}{3}E_1$$

and

$$D_1 = -\tfrac{1}{6}E_1 \qquad D_4 = -\tfrac{2}{3}E_1.$$

So, except for degenerate systems with

$$E_1 = F_3 = D_1 = D_4 \equiv 0$$

if one system (3.1) associated with (1.1) is a Lie system, then the other cannot be. It is therefore wrong to say that an equation (1.1), whose coefficients satisfy (1.10), is 'equivalent' to a Lie system defined by (3.1) and (3.17): it is also associated with a different system (3.1) with $D_1 \neq D_4$.

The relationship between (1.1) and two different systems (3.1) has interesting consequences. First, we can show that any solution $y(x)$ of (1.1) is a member of a solution pair $\{y(x), z_\pm(x)\}$ of either of the associated systems (3.1), which we write as

$$y' = A_1 + B_1 y + B_2 z_+ + D_1^+ y^2 \tag{3.18a}$$

$$z_+' = A_2 + B_3^+ y + B_4 z_+ + D_3^+ y^2 + D_4^+ y z_+ \tag{3.18b}$$

$$y' = A_1 + B_1 y + B_2 z_- + D_1^- y^2 \tag{3.18c}$$

$$z_-' = A_2 + B_3^- y + B_4 z_- + D_3^- y^2 + D_4^- y z \tag{3.18d}$$

where we note from (2.5) that taking alternative solutions for D_1 and D_4 gives alternative solutions for B_3 and D_3 but not for the other coefficients. Suppose that $y(x)$ satisfies (1.1); then define z_+ by (3.18*a*), remembering that $B_2 \neq 0$. This ensures that (y, z_+) satisfy (3.18*b*). Conversely, assuming (3.18*a*, *b*), we can substitute for z_+ from (3.18*a*) into (3.18*b*), obtaining (1.1). The same arguments hold for (3.18*c*, *d*). It is important to note that, in general, $D_1^+ \neq D_1^-$ and $D_4^+ \neq D_4^-$; then $z_+ \neq z_-$, although they correspond to the same 'partner function' $y(x)$.

This correspondence of solutions of (1.1), (3.18*a*, *b*) and (3.18*c*, *d*) can be useful for solving some more complicated equations. First, if we eliminate the function $y(x)$ from (3.18*a*, *b*), we usually obtain an equation for $z_+(x)$ of index 3. So if we can solve (1.1), we can use (3.18*a*) to give a solution of this index 3 equation. Second, if (3.18*a*, *b*) are a Lie system and therefore soluble in closed form, we know the function $y(x)$ satisfying (3.18*c*, *d*), which is *not* a Lie system. We can in fact express z_- in terms of z_+ and z'_+: subtracting (3.18*a*) from (3.18*c*)

$$(z_- - z_+)B_2 = (D_1^+ - D_1^-)y^2$$

and substituting in (3.18*b*) gives

$$z_- = z_+ + \frac{D_1^+ - D_1^-}{4(D_3^+)^2 B_2}\{-B_3^+ - D_4^+ z_+ \pm [(B_3^+ + D_4^+ z_+)^2 - 4(A_2 - z'_+ + B_4 z_+)D_3^+]^{1/2}\}^2. \tag{3.19}$$

When D_3 is sufficiently small, the positive square root in (3.19) has to be taken; for, as $D_3 \to 0$, the negative square root will tend to infinity instead of the correct solution

$$z_- = z_+ + \frac{D_1^+ - D_1^-}{B_2}\left(\frac{z'_+ - A_2 - B_4 z_+}{B_3^+ + D_4^+ z_+}\right)^2$$

when $D_3 = 0$.

Since (y, z_+) satisfy a Lie system, they obey a finite superposition principle. At first sight, the relationship (3.19) seems to contradict the fact that (y, z_-) do *not* obey a finite superposition principle. There is no contradition since z'_+ appears in (3.19). So any solution z_- can be written in the form (3.19), in which z_+ can be expressed in terms of a *finite* number of particular solutions (y, z_+) of (3.18*a*, *b*). So the general solution z_- *does* satisfy a different kind of finite superposition principle, in terms of solutions of (3.18*a*, *b*) and their derivatives.

One of the equations mentioned in the introduction was the bi-Riccati equation (1.5), equivalent to (1.1) with coefficients given by (1.6). By direct substitution, we see that these coefficients satisfy the conditions (1.10) for the existence of a corresponding Lie system. This system can be obtained by writing

$$z = y' + fy + gy^2$$

in (1.10), reducing the equation to

$$z' + fz + gyz + k + hy = 0.$$

This pair of first-order equations satisfies (3.17*b*), and so forms a Lie system. There is a converse to this property: any equation (1.1) which corresponds to a Lie system can be written in the form (1.5) and (1.6). To show this, we use the conditions (1.10) to express (1.1) as

$$y'' + (E_0 + E_1 y)y' + F_0 + F_1 y + \tfrac{1}{3}(E'_1 + E_0 E_1)y^2 + \tfrac{1}{9}E_1^2 y^3 = 0.$$

The coefficients in this equation will be exactly those given by (1.6) if we choose

$$f = \tfrac{1}{2}E_0 \qquad g = \tfrac{1}{3}E_1 \qquad k = F_0$$

$$k = F_1 - \tfrac{1}{2}E'_0 - \tfrac{1}{4}E_0^2.$$

We have therefore shown that every equation of form (1.1) associated with a first-order Lie system is equivalent to a bi-Riccati equation of form (1.5) and (1.6).

The first-order equations (3.1) for y and z can then be written in the matrix Riccati form (1.11). This is an example of a 'projective Riccati' equation as there is only one row in the matrix D. The general solution of this type of equation is a superposition of $n+2$ particular solutions, where n is the number of columns in D, equal to two in our study (Anderson 1980, Anderson *et al* 1981, 1982).

The matrix Riccati equation (1.11) may be linearised by introducing homogeneous coordinates in the standard way. Thus $W(x) \equiv U(x)V^{-1}(x)$ satisfies (1.11) if U, V are $\{1\times 2\}$ and $\{1\times 1\}$ matrices respectively, satisfying

$$U' = BU + AV \tag{3.20a}$$

$$V' = -DU - CV. \tag{3.20b}$$

We have chosen $C \equiv 0$; when $A \equiv 0$ also, (3.18) may be integrated formally to give the solution

$$U(x) = \exp\left(\int_0^x B(x')\,\mathrm{d}x'\right)U(0) \tag{3.21a}$$

$$V(x) = V(0) - \int_0^x \left[D(x')\exp\left(\int_0^{x'} B(x'')\,\mathrm{d}x''\right)U(0)\right]\mathrm{d}x'. \tag{3.21b}$$

In § 4, we use these formal integrals to study solutions of (3.20) when the coefficients in the corresponding equation (1.1) are constants.

4. Equations with constant coefficients

Let us first show how an equation (1.11), with constant coefficients and with $C=0$, may be transformed to an equivalent equation with $A=0$. Substitute $W(x) = \mathbf{W}(x) + P$ into (1.11) where

$$P = \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}$$

is a constant vector. Then $\mathbf{W}$ satisfies the equation

$$\mathbf{W}' = \mathbf{A} + \mathbf{B}\mathbf{W} + \mathbf{W}\mathbf{C} + \mathbf{W}\mathbf{D}\mathbf{W} \tag{4.1}$$

where

$$\mathbf{A} = \begin{pmatrix} A_1 \\ A_2 \end{pmatrix} + \begin{pmatrix} B_1 & B_2 \\ B_3 & B_4 \end{pmatrix}\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} + \begin{pmatrix} P_1 \\ P_2 \end{pmatrix}(D_1\ D_2)\begin{pmatrix} P_1 \\ P_2 \end{pmatrix} \tag{4.2a}$$

$$\mathbf{B} = \begin{pmatrix} B_1 + P_1 D_1 & B_2 + P_1 D_2 \\ B_3 + P_2 D_1 & B_4 + P_2 D_2 \end{pmatrix} \tag{4.2b}$$

$$\mathbf{C} = (D_1 P_1 + D_2 P_2) \qquad \mathbf{D} = (D_1\ D_2). \tag{4.2c}$$

The general matrix Riccati equation with variable coefficient matrices has been studied recently from the viewpoint of fundamental sets of solutions and superposition formulae (Harnad *et al* 1983, del Olmo *et al* 1987). Therefore $A \equiv 0$ if

$$A_1 + B_1 P_2 + B_2 P_2 + D_1 P_1^2 + D_2 P_1 P_2 = 0 \tag{4.3a}$$

$$A_2 + B_3 P_1 + B_4 P_2 + D_1 P_1 P_2 + D_2 P_2^2 = 0. \tag{4.3b}$$

We can eliminate P_2 to get a quartic equation for P_1. If P_1 is any root of this equation then P_2 may be chosen so that (4.3) are satisfied and hence $\mathbf{A}=0$. We can also absorb $\mathbf{C}$ into $\mathbf{B}$, as before.

We therefore assume that $\mathbf{A}=0$ for constant-coefficient equations (4.1) and the solutions are then of the form (3.21). Since the integrals in (3.21) with constant coefficients are trivial, the solution of (4.1) is

$$W(x)=\exp(\mathbf{B}x)(W(0)-P)\{\mathbf{1}-\mathbf{D}\mathbf{B}^{-1}[\exp(\mathbf{B}x)-\mathbf{1}]\}^{-1}+P \tag{4.4}$$

where we have taken $U(0)=W(0)-P$, $V(0)=\mathbf{1}$ and $\mathbf{C}=0$. The corresponding solutions for $y(x)$ and $z(x)$ have a simple form when the diagonal part of the matrix

$$\mathbf{B}=\begin{pmatrix} B_1' & B_2' \\ B_3' & B_4' \end{pmatrix}$$

commutes with the antidiagonal part. A sufficient condition for this to happen is that $B_1'=B_4'$; then if $B_2'B_3'=\lambda^2$ with λ real,

$$\exp(\mathbf{B}x)=\exp(B_1'x)\left[\mathbf{1}\cosh\lambda x+\lambda^{-1}\begin{pmatrix} 0 & B_2' \\ B_3' & 0 \end{pmatrix}\sinh\lambda x\right] \tag{4.5a}$$

while if $B_2'B_3'=-\lambda^2$, then

$$\exp(\mathbf{B}x)=\exp(B_1'x)\left[\mathbf{1}\cos\lambda x+\lambda^{-1}\begin{pmatrix} 0 & B_2' \\ B_3' & 0 \end{pmatrix}\sin\lambda x\right]. \tag{4.5b}$$

Substituting from (4.5) into (4.4), one obtains a solution of (3.1), subject to (3.17*b*), in terms of either hyperbolic or trigonometric functions of λx.

We shall use these methods to find explicit solutions for the particular equation (1.3) considered by Ervin *et al* (1984). We shall also illustrate how one would treat the general equation (1.1) with constant coefficients. For equation (1.3), the coefficients in (1.1) are

$$E_0=0 \qquad E_1=3\gamma+1 \qquad F_0=c \qquad F_1=F_2=0 \qquad F_3=3\gamma-1. \tag{4.6}$$

Choosing $A_1=0$, $B_1=b$, $B_2=1$ in (2.1), and using (2.6), (2.7) and (2.4*a*, *b*), we find that the two choices of D_1, D_4 are

$$D_4=-2 \qquad D_1=-\tfrac{1}{2}(3\gamma-1) \tag{4.7a}$$

and

$$D_4=-(3\gamma-1) \qquad D_1=-1 \tag{4.7b}$$

while

$$\begin{aligned} &B_4=-b \qquad A_2=-c \qquad B_3=-b^2 \\ &D_2=D_5=D_6=0 \qquad D_3=b(D_4-D_1). \end{aligned} \tag{4.8}$$

Ervin *et al* (1984) found exact solutions of their equation for three values of γ. One value was $\gamma=\frac{1}{3}$, for which (4.7) gives either $D_1=0$ or $D_4=0$. With these values for the coefficients and the choice $b=0$, equations (3.1) with (4.7*a*) become

$$y'=z \qquad z'=-c-2yz \tag{4.9a}$$

while (3.1) with (4.7*b*) give

$$y'=z-y^2 \qquad z'=-c. \tag{4.9b}$$

Both pairs of equations (4.9) have the integral

$$y' = -cx - y^2 + c_0$$

where c_0 is a constant of integration; this equation can be used to give the solution with $\gamma = \frac{1}{3}$ found by Ervin *et al.*

The conditions for a system (3.1) to be a Lie system are (3.17b). If $D_4 = D_1$, then (4.8) ensures that $D_3 = 0$, so that we only require $D_4 = D_1$. So the system defined by (4.7a) is a Lie system when $3\gamma - 1 = 4$, or $\gamma = \frac{5}{3}$; that defined by (4.7b) is a Lie system if $\gamma = \frac{2}{3}$. These are precisely the other two values of γ for which Ervin *et al* obtained solutions; for each value, the coefficients in (1.1) satisfy (1.10). We now show how their solutions can be obtained using the methods of this section.

Consider first the value $\gamma = \frac{5}{3}$. Substituting in (4.7a, b) we obtain the values $D_4 = D_1 = -2$ or $D_4 = -4$, $D_1 = -1$. This exemplifies the important fact that only one of the sets of first-order equations are of projective Riccati type, with $D_1 = D_4$. We shall discuss the 'non-Riccati' first-order system later.

Using the values given by (4.7a) and (4.8), equations (4.3) become

$$bP_1 + P_2 - 2P_1^2 = 0 \qquad -c - b^2P_1 - bP_2 - 2P_1P_2 = 0. \tag{4.10}$$

In the notation of Ervin *et al*, $c = \frac{1}{2}F^3$; then a solution of (4.10) is $P_1 = -\frac{1}{2}F$, $P_2 = \frac{1}{2}F(b+F)$. Substituting these values of P_1, P_2 and $D_1 = D_4 = -2$ in (4.8) and (4.2), we find

$$\mathbf{A} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \qquad \mathbf{B} = \begin{pmatrix} b+F & 1 \\ -b^2 - F^2 - bF & -b \end{pmatrix}$$

$$\mathbf{C} = (F) \qquad \mathbf{D} = (-2\ 0). \tag{4.11}$$

We now choose $b = -\frac{1}{2}F$, so that the diagonal elements of $\mathbf{B}$ are equal. The product of the antidiagonal elements is then $-\frac{3}{4}F^2 \equiv -\lambda^2$, say. The solution of (1.11) is then given by (4.4) and (4.5b) with $\lambda = \pm\frac{1}{2}\sqrt{3}F$ and with $\mathbf{C}$ absorbed' into $\mathbf{B}$ by taking $\mathbf{C} = 0$ and

$$\mathbf{B} = \begin{pmatrix} \frac{3}{2}F & 1 \\ -\frac{3}{4}F^2 & \frac{3}{2}F \end{pmatrix}. \tag{4.12}$$

After some tedious but elementary matrix algebra, one obtains the following solution for the first component of $W(t)$:

$$\begin{aligned} y(t) = P_1 &+ \exp(B_1''x)[(y(0) - P_1)\cos\lambda x + \lambda^{-1}(z(0) - P_2)\sin\lambda x] \\ &\times (1 - (B_1''^2 - B_2''B_3'')^{-1}\{D_1B_1''[(\exp(B_1''x)\cos\lambda x - 1)(y(0) - P_1) \\ &- \exp(B_1''x)\lambda^{-1}B_2''(z(0) - P_2)\sin\lambda x] \\ &- D_1B_2''[(\exp(B_1''x)\cos\lambda x - 1)(z(0) - P_2) \\ &+ \exp(B_1''x)\lambda^{-1}B_3''(y(0) - P_1)\sin\lambda x]\})^{-1} \end{aligned} \tag{4.13}$$

where B_i'' are the elements of $\mathbf{B}$ given in (4.12), $P_1 = -\frac{1}{2}F$, $P_2 = \frac{1}{4}F^2$ and $\lambda = \frac{1}{2}\sqrt{3}F$. The initial value $z(0)$ is obtained from (2.1a) using the coefficients given in (4.6) and (4.7)

$$z(0) = y'(0) - by(0) + 2y(0)^2. \tag{4.14}$$

Therefore (4.13) and (4.14) give the solution of (1.3) when $\gamma = \frac{5}{3}$ for prescribed values of $y(0)$, $y'(0)$. Comparing this with the solution given by Ervin *et al* (1984) in their equation (29), we see that our solution (4.13) has exactly the same functional form. Since the solutions also satisfy the same initial conditions, they must be identical.

Exactly the same procedure may be followed when $\gamma=\frac{2}{3}$. Then the solution (2.6) corresponding to the projective Riccati equation is $D_1=D_4=-1$ while the other solution is $D_4=2$, $D_1=-\frac{1}{2}$. Assuming $D_1=D_4=-1$ and substituting from (4.7), (4.3) becomes

$$bP_1+P_2-P_1^2=0 \qquad -\tfrac{1}{2}F^3-b^2P_1-bP_2-P_1P_2=0. \tag{4.15}$$

A solution of (4.15) is

$$P_1=-2^{-1/3}F \qquad P_2=2^{-2/3}F^2+2^{-1/3}bF. \tag{4.16}$$

The matrix coefficients in (4.1) become

$$\mathbf{A}=\begin{pmatrix}0\\0\end{pmatrix} \qquad \mathbf{B}=\begin{pmatrix} b+2^{-1/3}F & 1\\ -b^2-2^{-2/3}F^2-2^{-1/3}bf & -b\end{pmatrix}$$

$$\mathbf{C}=2^{-1/3}F \qquad \mathbf{D}=(-1\ 0). \tag{4.17}$$

The diagonal elements of $\mathbf{B}$ are equal when we take $b=-F/2^{4/3}$; then the product of the antidiagonal elements is $-3F^2/2^{8/3}\equiv-\lambda^2$. The solution of (1.11) is then given by (4.13) with $\lambda=3^{1/2}F/2^{4/3}$; P_1, P_2 are given by (4.16) and the B_i'' are the elements of $\mathbf{B}$ after 'absorption' of $\mathbf{C}$, which gives

$$\mathbf{B}=\begin{pmatrix} 3F/2^{4/3} & 1\\ -3F^2/2^{8/3} & 3F/2^{4/3}\end{pmatrix}. \tag{4.18}$$

The initial value of $z(x)$ given by (2.1a) is now

$$z(0)=y'(0)-by(0)+y(0)^2. \tag{4.19}$$

The solution is again identical to that given by Ervin *et al* (1984), now with $\gamma=\frac{2}{3}$.

5. A Hamiltonian system

We have seen that an equation of form (1.1) generally corresponds to two sets of equations of form (3.1). In particular, when one set is a Lie system, the other is not. However, some equations of form (1.1) can be derived from a Lagrangian and we are accustomed to a one-to-one correspondence between second-order Lagrangian equations and first-order Hamiltonian equations. In this section, by examining a simple example, we shall show how this apparent contradiction is resolved.

The Lagrangian

$$L=\tfrac{1}{2}y'^2+\tfrac{1}{2}y^2+\tfrac{1}{3}\mu y^3+\tfrac{1}{4}\lambda^2y^4 \tag{5.1}$$

with λ and μ constant, gives rise to the second-order equation

$$y''-y-\mu y^2-\lambda^2y^3=0 \tag{5.2}$$

of the form (1.1). When $\mu=0$, (5.2) is of the form (1.2), and is invariant under the transformation $y\to-y$. The corresponding first-order Riccati type systems are

$$y'=z+D_1y^2 \tag{5.3a}$$

$$z'=y+\mu y^2+D_4yz \tag{5.3b}$$

where

$$D_1=\pm2^{-1/2}\lambda \qquad D_4=\mp2^{1/2}\lambda. \tag{5.4}$$

The Hamiltonian corresponding to (5.1) is

$$H = \tfrac{1}{2}p^2 - \tfrac{1}{2}y^2 - \tfrac{1}{3}\mu y^3 - \tfrac{1}{4}\lambda^2 y^4 \tag{5.5}$$

giving rise to the Hamiltonian equations

$$y' = p \tag{5.6a}$$

$$p' = y + \mu y^2 + \lambda^2 y^3. \tag{5.6b}$$

It is easy to check that these canonical equations are compatible with (5.3) for both choices of sign in (5.4). This is because λ appears in (5.6) only in the form λ^2.

For given (y, z), the velocity vector (y', z') depends upon the choice of sign in (5.4). So the different signs give different sets of trajectories in the (y, z) plane; when $\mu = 0$, the two sets of trajectories are related by the central inversion $(y, z) \to (-y, -z)$. But in the (y, p) phase plane, (5.6) define a unique set of trajectories.

Therefore, if we question whether there are one or two sets of trajectories for a Hamiltonian system giving rise to an equation of form (1.1), then the answer is that it depends on the choice of variables used to describe the system.

6. Conclusions

We have considered here the factorisation of the second-order non-linear differential equation (1.1) into a pair of coupled first-order equations (1.9) of Riccati type. Our results may be summarised as follows.

(*a*) The factorisation into coupled equations of the form (1.9*a*, *b*) with $D_2 \equiv D_5 \equiv D_6 \equiv 0$ is always possible.

(*b*) There are then three further coefficients which may be chosen arbitrarily; it was found most convenient to take these to be A_1, B_1, B_2, with $B_2 \neq 0$.

(*c*) There remains a twofold choice of first-order equations corresponding to the two sets of values for D_1, D_4 given by (2.6). The different choices do not give new solutions of (1.1), but give different partner functions z to y related by (3.19).

(*d*) Conditions (1.10) on the coefficients of (1.1) ensure that the corresponding first-order equations are of Lie type. When these equations are both non-linear and do not decouple, the associated Lie algebra is sl(3, R). The related second-order equation is then of bi-Riccati form (1.5).

(*e*) The first-order equations of Lie type can be written in matrix Riccati form and integrated formally in the standard way. Consequently for constant-coefficient equations (1.1), solutions are obtained in terms of trigonometric or hyperbolic functions. For the particular equation (1.3), these reduce to the solutions found by Ervin *et al* (1984) when $\lambda = \frac{5}{3}$ or $\frac{2}{3}$.

(*f*) Examples of equations (1.1) derivable from a Lagrangian have been discussed and a comparison made between the corresponding first-order equations (1.9) and the Hamiltonian equations which are also equivalent to (1.1).

Conditions (1.10) determine whether (1.1) can be reduced to a first-order Lie system. The system is then quite well understood, and (1.1) is of bi-Riccati form. When the Lie conditions are not satisfied, the corresponding first-order systems are much less manageable, and there exists no general mathematical method for finding and classifying their solutions.

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